2. Theory of Small Amplitude Waves

2.1 General Discussion on Waves

Let us consider a one-dimensional (on x-axis) propagating wave that retains its original shape. Assume that the wave can be expressed as a function at an initial time t = 0 as follows:

$$\eta = f(x) \qquad (1)$$

If this wave shifts to the right with a velocity V, then it takes the form expressed as

$$\eta = f(x - Vt) \qquad (2)$$

where t is time.

(Note: In high school, you must have studied that if a hyperbolic function $y = x^2$ is shifted to the right by 7, then it takes the form $y = (x-7)^2$. In general, the position of the original graph moves to the right by a, when we change x into $x - a$.)

We substitute the subordinate variable $x - Vt = \xi$ in the right-hand side of (2); therefore, $\eta = f(\xi)$. We differentiate η partially with x, thereby obtaining

$$\partial\eta/\partial x = d\eta/d\xi \times \partial\xi/\partial x = d\eta/d\xi \qquad (3)$$

Further, when we differentiate it partially by t, we obtain

$$\partial\eta/\partial t = d\eta/d\xi \times \partial\xi/\partial t = -V\, d\eta/d\xi \qquad (4)$$

Comparing (3) and (4), and eliminating $d\eta/d\xi$, we have the following partial differential equation:

$$\frac{\partial\eta}{\partial t} = -V\frac{\partial\eta}{\partial x} \quad \text{or} \quad \frac{\partial\eta}{\partial t} + V\frac{\partial\eta}{\partial x} = 0 \qquad (5)$$

Equations (5) and (2) are mathematically equivalent to each other, and both equations are expressions of a wave moving one dimensionally toward the right on the x-axis at a velocity V without changing its form.

2.2 Wave Equation

Equation (2) (or (5)) is a formulation of a wave propagating in the positive x direction (i. e., to the right). In general cases, there are two wave components—one moving toward the left and the other toward the right, which have the same wave velocity and overlap. For such general cases, the equation of a wave is expressed as the sum of the two wave components. Hence, in place of equation (2), we must introduce the following form as a general wave with these two components:

$$\eta = f(x - Vt) + g(x + Vt) \qquad (6)$$

where f, g are arbitrary functions giving the wave form. By differentiating (6), first with respect to x and then with t, and eliminating the arbitrary functions f, g, we have the following second-order partial differential equation.

$$\frac{\partial^2 \eta}{\partial t^2} = V^2 \frac{\partial^2 \eta}{\partial x^2} \qquad (7)$$

Equation (7) is mathematically equivalent to (6), and it is called the "wave equation" of the one-dimensional case. The wave velocity is often written as "c" instead of "V."

[Mathematical Note]

Solution of a partial differential equation of two independent variables:

We introduce D_x and D_t as the operators of partial differentiation in x direction and time t, that is,

$$D_x = \partial/\partial x, D_t = \partial/\partial t \qquad (8)$$

and z is assumed to be a subordinate function of x and t, that is,

$$z = f(x,t)$$

[Problem 1] Solve the following partial differential equation:

$$\frac{\partial^2 z}{\partial x^2} - \frac{\partial^2 z}{\partial x \partial t} - 6\frac{\partial^2 z}{\partial t^2} = 0 \qquad (9)$$

[Answer] We re-write (9) by using (8); then we have

$$(D_x^2 - D_x D_t - 6D_t^2)z = 0 \qquad (10)$$

We can reduce (10) into the following form by using the method of resolution of factors :

$$(D_x + 2D_t)(D_x - 3D_t)z = 0 \qquad (11)$$

Here, the a table that shows "the re-writing rule" is prepared as follows:

$$D_x \Rightarrow t, D_t \Rightarrow x, + \Rightarrow -, - \Rightarrow +$$

Thus, we obtain the solution of (9) as follows:

$$z = f(t - 2x) + g(t + 3x) \qquad (12)$$

where f and g are arbitrary differential functions.

By using the result given in the mathematical note, we can obtain the solution of the equation of wave (7) in the following form:

$$\eta = f(x - Vt) + g(x + Vt) \qquad (13)$$

This is nothing but equation (6).

Wave equations of two- and three-dimensional problems

For two-dimensional waves such as an ocean waves, the wave equation takes the following form:

$$\frac{\partial^2\eta}{\partial t^2}=V^2\left(\frac{\partial^2\eta}{\partial x^2}+\frac{\partial^2\eta}{\partial y^2}\right) \qquad (14)$$

For three-dimensional cases such as seismic P and S waves, electro-magnetic waves, and sound waves, the wave equation takes the following form:

$$\frac{\partial^2\eta}{\partial t^2}=V^2\left(\frac{\partial^2\eta}{\partial x^2}+\frac{\partial^2\eta}{\partial y^2}+\frac{\partial^2\eta}{\partial z^2}\right) \qquad (15)$$

Here, the right-hand side has the Laplacian operator.

In any physical problem, if a subordinate variable satisfies equation (10), it can be shown that the value of the subordinate variable propagates as a wave having velocity V.

2.3 Periodic Waves

When a wave shape takes the form of a periodic repetition of the same pattern, we call the repeating interval as a "wavelength L". If you watch any point on the x-axis, we observe the same variation in a uniform time interval; this time interval called the "wave period T". Similar to the case of a train with a length L traveling with speed V, the period T is given by

$$T = L/V \quad \text{or} \quad TV = L \qquad (16\text{-a,b})$$

Instead of the wavelength L, we often use "wave number k," which is defined as

$$k = 2\pi / L$$

and "angular frequency σ" defined as

$$\sigma = 2\pi / T$$

It is possible to rewrite (16-a,b)as

$\sigma = ck$ When the wave shape consists of simple harmonics , it can be expressed as

$$\eta = a\sin(kx - \sigma t) = a\sin(2\pi x / L - 2\pi t / T)$$

where "a" is called "amplitude ."By introducing an expression in a complex function, we have

$\eta = -iae^{i(kx-\sigma t)}$ from the famous relationship

$$e^{ix} = \cos x + i\sin x$$

2.4 Ocean Wave Theory of Infinite Amplitude

We place the origin at a point on a stable sea surface, and take the x-axis horizontally in the wave propagation direction, while the y-axis is assumed to be vertically upward. The ocean depth is a constant D (m). We neglect the influence of the rotation of the

Earth. We assume that sea water is a perfect fluid (no viscosity) and has non-vortex motion; then we can introduce the velocity potential function ϕ.

The equation of mass conservation is given in the following form:

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0 \qquad (17)$$

Now, we consider a wave, i. e, a periodically repeating phenomenon in the x-direction. If the wavelength is considered to be L, we can introduce the form of ϕ in the following form:

$$\phi = F(y)\cos\{k(x-ct)\} \qquad (18)$$

This express the motion pattern in the form of a sinusoidal wave with a wave number k $(=2\pi/L)$ and velocity c. The wave number k is related to the length L in the following manner:

$$k = 2\pi/L \qquad (19)$$

We substitute $kc = \sigma$. This is called angular frequency (radian/s), and it is related to the period T as follows:

$$\sigma = 2\pi/T \qquad (20)$$

If we introduce the expression by using complex numbers, then

$$\phi = F(y)e^{i(kx-\sigma t)} \qquad (21)$$

Hereafter, we use the complex expression given in (21). The real part of the equation will give the form of the actual (visible) form . You must be aware of the following formula

$$e^{ix} = \cos x + i\sin x, \quad e^{i(x+y)} = e^{ix} \times e^{iy}, \ \left(e^{ix}\right)' = ie^{ix}$$

Substituting (21) into (17) we have

$$F'' - k^2 F = 0 \qquad (22)$$

This is a second-order linear differential equation with constant coefficients, and its solution has the following form:

$$F = C_1 e^{ky} + C_2 e^{-ky} \qquad (23)$$

Finally, we have the form of the velocity potential as

$$\phi = \left(C_1 e^{ky} + C_2 e^{-ky}\right)e^{i(kx-\sigma t)} \qquad (24)$$

[Sea bottom condition]

If we assume that the ocean depth is constant $(=D)$, there is no vertical component of the water particle velocity on the ocean bed.

$$v = 0 \quad \text{at} \quad y = -D \qquad (25)$$

We write this condition by using φ as follows:

$$\left[\frac{\partial \phi}{\partial y}=0\right]_{y=-D} \text{; hence, } C_1 e^{-kD}=C_2 e^{kD} \qquad (26)$$

If we put both side value of (26) as $C/2$ then we have

$$\phi = C\cosh k(y+D)e^{i(kx-\sigma t)} \qquad (27)$$

Note: “$\cosh$” is “hyperbolic cosine” and is defined as $\cosh x \equiv \dfrac{e^x+e^{-x}}{2}$. Similarly, $\sinh, \tanh$ refer to “hyperbolic sine” and “hyperbolic tangent,” and they are defined as $\sinh x = \dfrac{e^x - e^{-x}}{2}$ and $\tanh(x)=\sinh x/\cosh x$, respectively. They have many relationships among each other and for differentiation similarly to trigonometric functions such as sin, cos, and tan. The differentiations are given by $(\cosh x)' = \sinh x, (\sinh x)' = \cosh x$.

[Sea surface boundary condition]

(1) Kinematic boundary condition

Since the present problem is a two-dimensional one (x, y), the kinematic boundary condition on the sea surface is given by

$$v=\frac{\partial \eta}{\partial t}+u\frac{\partial \eta}{\partial x}, \quad \text{at} \quad y=\eta \qquad (28)$$

Here we introduce “the order of small value.” We define that the values (u, v, ϕ, η) <□>, which are induced by the wave and are linearly correlated with each other are small values of the order of $O(\varepsilon^1)$. The product of any two elements of <□> are the values of the second-order small values $O(\varepsilon^2)$ In the scope of “linear theory” we assume that we can neglect values that have a higher order than 2 on comparison with the first-order ones.

Equation (28) is the condition on the surface $y=\eta$, and the v-value on the left-hand side is written in the form of a Taylor series as follows:

$$[v]_{y=\eta} = [v]_{y=0} + \eta\left[\frac{\partial v}{\partial y}\right]_{y=0} + \cdot\cdot\cdot \qquad (29)$$

Hence, keeping in mind the scope of the linear theory, we can neglect the second- and higher-order terms.

The second term on the right-hand side of (28) is obviously a higher-order value of the first term. Hence, (28) can be approximately re-written in the following form:

$$[v]_{y=0} = \frac{\partial \eta}{\partial t} \qquad (30)$$

(2) Dynamic boundary condition on the sea surface

This condition is derived from the fact that the pressure is uniformly equal to the atmospheric pressure. If we assume the motion is a non-vortex one (□), we can introduce a velocity potential function φ, and Bernoulli's formula should be satisfied.

$$-\frac{\partial \phi}{\partial t} + \frac{1}{2}(u^2 + v^2) + g\eta + p/\rho = F(t) \qquad \text{at} \quad y = \eta \qquad (31)$$

We neglect second-order terms and assume that the water pressure is equal to the atmospheric pressure; (31)then becomes

$$-\frac{\partial \phi}{\partial t} + g\eta + P_0/\rho = F(t) \qquad \text{at} \quad y = 0 \qquad (32)$$

We partially differentiate (32) with respect to t, and substitute (31); then we have

$$\frac{\partial^2 \phi}{\partial t^2} = g\frac{\partial \phi}{\partial y} \qquad \text{at} \quad y = 0 \quad (33)$$

(Note: There is an arbitrary function of time, F(t), in the right-hand side of (32), which implies that it is permissible to change atmospheric pressure; however, this is neglected in the present case)

Substituting (27) in (33) gives the relationship between the wave number k and angular frequency σ as

$$\sigma^2 = gk \tanh kD \qquad (34)$$ <Dispersion Relation >

Since $\sigma = ck$ (c : wave velocity, phase velocityfrom (16), (19), and (20),

$$c = \sqrt{\frac{g}{k}\tanh kD} \qquad (35)$$

or

$$c = \sqrt{\frac{gL}{2\pi}\tanh\frac{2\pi D}{L}} \qquad (36)$$

This is the formula for wave velocity c given depth of water D and wavelength L.

Since the phase velocity changes with the wavelength, the wave shape cannot always maintain its initial form, and we call such waves "dispersive waves."

[Relationship between wave amplitude and water particle velocity]

We assume that the water surface takes a sinusoidal form as follows:

$$\eta = a\sin(kx - \sigma t) \qquad \text{or} \qquad \eta = -iae^{i(kx-\sigma t)} \quad (37)$$

where a denotes the amplitude, and $2a$ is the wave height measured from the bottom of the trough to the top of the wave crest.

Substituting (37) and (30) in (31b), C in (27) can be expressed by the amplitude, that is,

$$C = \frac{ac}{\sinh kD} \qquad (4\text{-}28)$$

Thus, the velocity potential function can be obtained in the following form:

$$\phi = \frac{ac}{\sinh kD} \cosh k(y+D) e^{i(kx-\sigma t)} \qquad (4\text{-}29)$$

We can calculate the distribution of water particle motion (u,v) by differentiating partially in the x, and y directions.

[Classification of waves]

We return to the velocity formula (36)

$$c = \sqrt{\frac{gL}{2\pi} \tanh \frac{2\pi D}{L}} \qquad (36)$$

Here the function $\tanh x$ can be approximated as $\tanh x \cong 1$ when x is larger than 3. On the other hand, it can also be approximated as $\tanh x \cong x$ when x is smaller than 0.3. Thus we can classify waves into three types, according to the value x = $2\pi D/L$.

(1) **Long wave** When $L > 20D$, x= $(2\pi D/L) < 0.3$ .Therefore, we approximate $\tanh(2\pi D/L) \cong 2\pi D/L$, making (39)

$$c = \sqrt{gD} \qquad (40)$$

Since the phase velocity is not dependent on the wavelength L, no dispersion is observed for this wave.

(2) Shallow water wave When $2D < L < 20D$. We cannot apply any approximation. Instead, we only use (36) itself.

(3) Deep water wave: When $L < 2D$ (i..e, the wavelength is smaller than half the water depth.)

Since x = $(2\pi D/L) > 3.0$, we can approximate $\tanh(2\pi D/L) \cong 1$, and (36) becomes $\sigma^2 = gk$. The phase velocity is given by

$$c = \sqrt{\frac{g}{k}} = \sqrt{\frac{gL}{2\pi}} \qquad (41)$$

Naturally, a dispersive wave is present in this case, and the wave velocity does not depend on the depth D .

[Orbit of water particles]

The velocity field of water particles $\vec{u}$ (u,v) can be calculated by differentiating the velocity potential function (39) in the x, y directions as follows:

$$u = -ia\sigma \frac{\cosh k(y+D)}{\sinh kD} e^{i(kx-\sigma t)} \qquad (42a)$$

$$v = a\sigma \frac{\sinh k(y+D)}{\sinh kD} e^{i(kx-\sigma t)} \qquad (42b)$$

By comparing (42a) and (37), we note that both η and u are multiplied by $-i$; hence, they change in the same phase with each other. By integrating u, v with respect to time t, we obtain the orbit of a water particle at the position (x, y) as follows:

$$x = a\frac{\cosh k(y+D)}{\sinh kD}\cos(kx-\sigma t), \quad y = a\frac{\sinh k(y+D)}{\sinh kD}\sin(kx-\sigma t) \qquad (43a,b)$$

This means that each water particle moves on the orbit of an ellipsoid with a center $(\bar{x}, \bar{y})$.

$$\frac{(x-\bar{x})^2}{\left[a\dfrac{\cosh k(\bar{y}+D)}{\sinh kD}\right]^2} + \frac{(y-\bar{y})^2}{\left[a\dfrac{\sinh k(\bar{y}+D)}{\sinh kD}\right]^2} = 1 \qquad (44)$$

In the case of deep water waves, (44) becomes

$$(x-\bar{x})^2 + (y-\bar{y})^2 = (ae^{k\bar{y}})^2 \qquad (45)$$

This means that the orbit is a circle.

3 Filter characteristics of a tsunami gauge at sea bed

The Japan Meteorological Agency (Kishocho) has installed pressure gauges on the sea bed in the sea off Tokai District and off Boso Peninsula. Now, let us consider whether such pressure gauges can be used to detect tsunamis.

Neglecting the non-linear and constant terms in Bernoulli's formula gives

$$p = \rho\frac{\partial\phi}{\partial t} \qquad (46)$$

Substituting the velocity potential φ and taking $y = -D$, we obtain

$$p = -i\rho a k^{-1}\sigma^2 \frac{1}{\sinh kD} e^{i(kx-\sigma t)} \qquad (47)$$

By using the sea surface shape (37b) and dispersion relation (34), we can calculate the water pressure at the seabed p as follows:

$$p = \rho g \eta \frac{1}{\cosh kD} \qquad (48)$$

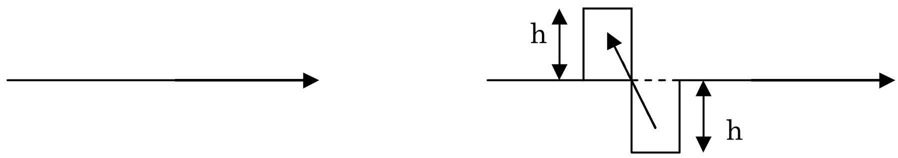
We expect the pressure gauge at the sea bed to reveal exact variations in depth $p = \rho g \eta$ in real time. However, in contrast of our expectations, (48) shows that there is an attenuation factor $1/\cosh kD$, which becomes multiplied on the pressure data.

JMA set four tsunami sensors at depths $D = 1912 - 4011\,\mathrm{m}$ in the Tokai and Boso regions. Okada (1991) proposed a diagram showing the relationship between the period T (s) and the attenuation factor $1/\cosh kD$. We note that a wave component with a period less than 2 min cannot be observed by such a type of sea bed pressure.

Since the period of a tsunami wave is generally 5 to 40 minutes, those pressure gauges function as a detector of tsunamis, but not as a detector of wind waves and swells.

4. Energy in an Infinite Amplitude Wave

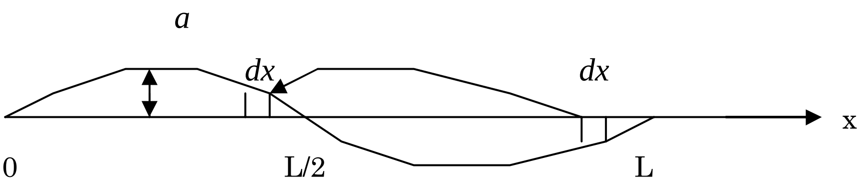
Next, let us consider the total energy contained in one wavelength of an infinite wave. There are two types of energy: potential energy T and kinematic energy V.



(A)

(B)

First, let us consider the potential energy difference between conditions (A) and (B) shown in the figure. The potential energy is given by $mg\Delta h$, where m is mass and Δh is the height difference. In the present case, since $m = \rho h dx$ and Δh = h, the potential energy dE of the unit cube is given by $dT = \rho g h^2 dx$.



If the wave shape is given by $\eta = a \sin kx; k = 2\pi / L$, the total potential energy is calculated by the following integral:

$$T = \int_0^{L/2} \rho g \eta^2 dx = \rho g \int_0^{L/2} a^2 \sin^2 kx dx = \frac{1}{4} \rho g a^2 L \qquad (49)$$

Next, we consider the kinematic energy V :

$$V = \frac{1}{2}\rho\int_0^L\int_{-D}^{\eta}\left(u^2+v^2\right)dydx = \frac{1}{2}\rho\int_0^L\int_{-D}^{\eta}\left\{\left(\frac{\partial\phi}{\partial x}\right)^2+\left(\frac{\partial\phi}{\partial y}\right)^2\right\}dydx \qquad (50)$$

Green's Law for two-dimensional cases is given as follows:

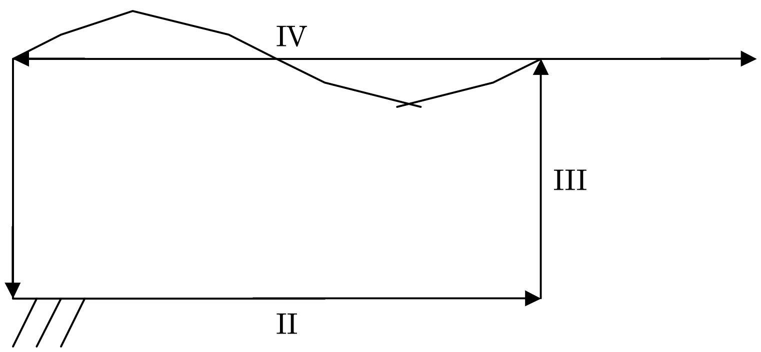
(Mathematical Note: Green's Law)

If ϕ is a harmonic function ($\nabla^2\phi = 0$), then

$$\iint_S\left(\phi_x^2+\phi_y^2\right)dS = \oint_l \phi\frac{\partial\phi}{\partial n}dl \qquad (51)$$

is satisfied, where n is an outward vector.

In the present case, the integral should be performed along the paths shown in the following figure I→II→III→IV.



The integral along II is obviously zero since $\partial\phi/\partial n = -\partial\phi/\partial y = [v]_{y=-D} = 0$.

The integrated functions on I and III are identical and the direction of the integral paths are opposite to each other; hence, the integrals along I and III cancel each other.

Thus, the kinematic energy in one wavelength is given by

$$V = \frac{1}{2}\rho\int_{IV}\phi\frac{\partial\phi}{\partial y}dl \qquad (52)$$

On the other hand, the kinematic sea surface condition is

$-\dfrac{\partial\phi}{\partial y} = v = \eta_t$ Hence, we have

$$V = -\frac{1}{2}\rho\int_{IV}\phi\eta_t dl = \frac{1}{2}\rho\int_0^L\phi_t\eta dx \qquad (53)$$

By using the linearized Bernoulli's formula ($-\phi_t + g\eta = 0$), we have

$$V = \frac{1}{2}\rho\int_0^L g\eta\times\eta dx = \frac{1}{4}\rho g a^2 L \qquad (54)$$

Thus, we have proved that the total energy E is divided equally between the potential and kinematic energy components.